Limits of the Tomimatsu–Sato gravitational field

William Kinnersley

Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91109

Edward F. Kelley

Department of Physics, Montana State University, Bozeman, Montana 59715

(Received 1 May 1974)

The Tomimatsu–Sato (TS) solutions of the Einstein field equations are studied in several limiting cases. In the weak-field limit we construct two Newtonian models for the source, one consisting of a rotating disc of radius a/n, the other made up of n complex point multipoles. The "extreme" limit $q=1$ is also examined in detail, and we find there are many distinct ways of taking this limit. We are thereby led to a new two-parameter family of exact solutions which, unlike the TS metrics, are not asymptotically flat.

I. INTRODUCTION

Recently Tomimatsu and Sato[1] have found a series of exact solutions of the Einstein vacuum field equations which they claim are suitable to represent the gravitational field of a rotating body. This would only be the second time such a solution has been discovered, the previous example being the Kerr metric. Further investigations by Glass[2] and by Gibbons and Russell-Clark[3] have shown that the TS solutions contain a naked singularity outside their event horizons and are therefore not black holes. The current popularity of black holes is so great that many persons would automatically reject a solution on this basis alone. However, there are perfectly good reasons for studying solutions that contain naked singularities. One whose conscience is troubled by them on astrophysical grounds may regard them as primordial remnants of the big bang, or else imagine an appropriate interior solution covering the region that would otherwise be offensive. At any rate we feel that the TS solutions have considerable mathematical and physical interest and deserve a great deal of further study.

In this paper we first try to understand the structure of the sources necessary to produce the TS field. We examine a weak-field limit in which the source has vanishingly small rest mass but finite size. In terms of the TS parameters, this implies the limit must be taken as $p, q\to\infty$. The linearized gravitational field obtained in this manner is exhibited in terms of a complex Newtonian potential. To understand its nature at large radial distances, we analyze the field into multipole moments. Near the origin, its singularities determine the source in terms of mass and mass-current distributions. The Newtonian models we thus obtain for the TS metrics are rotating discs qualitatively similar to the model discussed by Israel[4] for the Kerr metric but with a smaller radius, a/n. We also give a simpler but more abstract model, in which the source is represented as a small number of point multipoles, all located a complex distance along the axis of symmetry.

The limit $p=0$, $q=1$ is another case of particular interest which we examine in detail. We find that this limit is not unique and that different metrics result from taking it in various ways. The metrics we obtain in this manner are new rotating solutions, considerably simpler than TS. In this limit we find that the TS parameter n need no longer be restricted to integer values. Unfortunately, the new solutions are not asymptotically flat.

II. ERNST POTENTIAL

The line elements for the TS solutions are quoted in the Weyl—Papapetrou canonical form for stationary axisymmetric fields:

$$ds^2=f(dt-\omega d\varphi)^2-f^{-1}(e^{2\gamma}[d\rho^2+dz^2]+\rho^2 d\varphi^2). \tag{1}$$

Here ρ, φ, z are to be thought of as cylindrical coordinates in a flat 3-space, which we call the "Weyl space," and f, ω, γ are the field variables which depend only on ρ, z. Rather than work directly with these metric components, we find it convenient to follow other authors and focus attention on a quantity called the Ernst potential.[5]

The Ernst potential ξ is a complex scalar field. It is related to the metric by[6]

$$\mathrm{Re}\left(\frac{1-\xi}{1+\xi}\right)=f, \tag{2}$$

$$\nabla\mathrm{Im}\left(\frac{1-\xi}{1+\xi}\right)=\rho^{-1}f^2\mathbf{e}_\varphi\times\nabla\omega, \tag{3}$$

where ∇ is the gradient operator on the Weyl space and $\mathbf{e}_\varphi$ is a unit vector in the φ direction. In this formulation the Einstein equations accomplish three things for us: (i) They insure that Eq. (3) is integrable for ω, (ii) they tell us how to construct γ once f, ω are given, and (ii) they provide an equation which ξ must satisfy

$$(\xi\xi^*-1)\nabla^2\xi=2\xi^*\nabla\xi\cdot\nabla\xi. \tag{4}$$

If we write ξ in terms of its real and imaginary parts

$$\xi=\Phi+i\Omega,$$

then in the weak-field limit we have from Eqs. (2), (4),

$$f=1-2\Phi,\quad \nabla^2\Phi=0,\quad \nabla^2\Omega=0.$$

Hence the real part Φ of the Ernst potential becomes the ordinary Newtonian potential. The imaginary part Ω plays the role of a "magnetic" scalar potential in analogy with electrodynamics. This can be seen from the weak-field limit of the geodesic equations, where we find that the acceleration of a slowly moving test particle is

$$\mathbf{a}=-\nabla\Phi+\mathbf{v}\times\nabla\Omega.$$

Even in the exact theory, the Ernst potential may

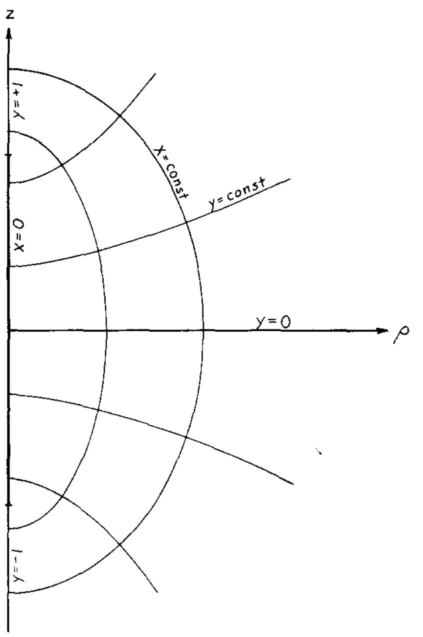

FIG. 1. Prolate spheroidal coordinates. The level surfaces $x=\text{const}$ are an orthogonal family of ellipsoids and hyperboloids. The semimajor axis of an ellipsoid is given by mpx/n, and the asymptotic angle of inclination of a hyperboloid is $\theta=\cos^{-1}y$.

still be conveniently regarded as a complexified nonlinear version of the Newtonian potential. We therefore feel it appropriate to concentrate on ξ as the quantity of direct physical and mathematical significance.

III. TS SOLUTIONS

The TS solutions to Eq. (4) contain arbitrary constant parameters $m, n, p, q,$ where $m>0$ is the mass, n is a positive integer, and p, q are dimensionless numbers related by

$$p^2+q^2=1. \tag{5}$$

For the first two values $n=1, 2$ the solutions are[1]

$$\xi_1^{-1}=px-iqy, \tag{6}$$

$$\xi_2^{-1}=\frac{p^2(x^4-1)-2ipqxy(x^2-y^2)-q^2(1-y^4)}{2px(x^2-1)-2iqy(1-y^2)} \tag{7}$$

Here x, y are prolate spheroidal coordinates in the Weyl space (see Fig. 1) related to ρ, z by

$$\begin{aligned}\rho&=(mp/n)(x^2-1)^{1/2}(1-y^2)^{1/2},\\ z&=(mp/n)\,xy.\end{aligned} \tag{8}$$

The cases $n=3, 4$ are also given in Ref. 1. The case $n=1$ is equivalent to the Kerr metric as discussed by Ernst.[5] For $n\geqslant 5$ the TS solutions have not been calculated but are presumed to exist. Since the $n=4$ solution already fills half a page, there is little incentive to pursue the matter further unless a general form for all n can be discovered. Charged TS solutions could also be written down, but since the procedure for doing this is now completely automatic and understood,[7] we feel that doing so would be a definite waste of time.

If we make the natural assumption that p is real, then Eq. (5) restricts q to the range $|q|\leqslant 1$. As TS themselves point out,[1] the solutions may easily be extended beyond this range via a complex coordinate transformation. We let

$$\hat{p}=-ip, \quad \hat{x}=ix \tag{9}$$

and assume instead that $\hat{p}, \hat{x}$ are the quantities which are real. Then we note the following facts:

(i) Since ξ contains p, x only in quadratic combinations, none of its terms lose their reality. Thus the meaning of ξ^* is unaltered, and we still have a solution of Eq. (4).

(ii) The relation between $\hat{p}, q$ is

$$q^2=\hat{p}^2+1, \tag{10}$$

and the restriction on q is now just the opposite of what it was before, namely $|q|\geqslant 1$.

(iii) The new coordinates $(\hat{x}, y)$ are *oblate* spheroidal coordinates in the Weyl space (see Fig. 2) and Eq. (8) is replaced by

$$\begin{aligned}\rho&=(m\hat{p}/n)(\hat{x}^2+1)^{1/2}(1-y^2)^{1/2},\\ z&=(m\hat{p}/n)\,\hat{x}y.\end{aligned} \tag{11}$$

Finally we note that, in all of the cases discussed above, $\xi\to 0$ as $\hat{x}$ or $x\to\infty$, and hence the solutions are all asymptotically flat.

IV. WEAK FIELD LIMIT

The exact mass, angular momentum, and quadrupole moment for the TS solutions have been given by Tomimatsu and Sato[1]:

$$M=m, \quad J=m^2q, \quad Q=m^3\left(\frac{n^2-1}{3n^2}p^2+q^2\right). \tag{12}$$

In the weak field limit as $m\to 0$ we see that J will be only $O(m^2)$, too small to survive, unless $q\to\infty$ at the same time. We therefore need to use the extended TS solutions. We define a Kerr parameter a by the equations

$$q=a/m, \quad p=(a^2-m^2)^{1/2}/m \tag{13}$$

and take the weak-field limit holding a finite. Just as in the Kerr metric itself, the parameter a has the dimensions of length and serves to describe the linear extent of the source.

Carrying out the stated limit on Eqs. (6), (7), we obtain for $n=1, 2$,

$$\xi_1=(m/a)X^{-1}, \tag{14}$$

$$\xi_2=(2m/a)(X^{-1}+i(\hat{x}y-i)X^{-3}, \tag{15}$$

where

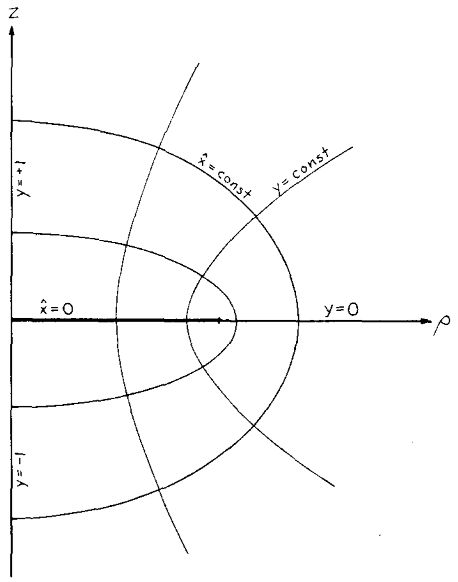

FIG. 2. Oblate spheroidal coordinates, Now $m\hat{p}\hat{x}/n$ specifies the semiminor axis of the ellipsoids.

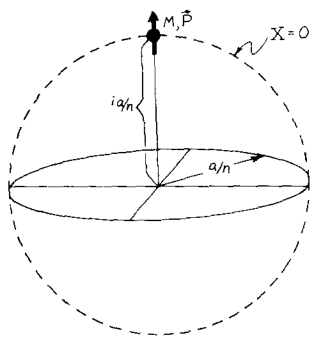

FIG. 3. Placement of point multipoles along the imaginary z-axis for the model discussed in Sec. IV.

$$X \equiv \hat{x} - iy. \tag{16}$$

Similarly from the Ernst potential for $n=3$ given in TS we obtain

$$\xi_3 = (3m/a)[X^{-1} + 2i(\hat{x}y - i)X^{-3} - \tfrac{2}{3}(3\hat{x}^2y^2 - \hat{x}^2 - 4i\hat{x}y + \hat{y}^2 - 3)X^{-5}]. \tag{17}$$

Equations (14), (15), (17) are all complex solutions of Laplace's equation, nonsingular everywhere except at $\hat{x}=y=0$. According to Eq. (11) this is the locus of a ring with radius a/n, so that whatever source is producing the field should reside there.

The results may be written in terms of spherical coordinates r, θ given by

$$\rho = r\sin\theta, \quad z = r\cos\theta$$

or alternatively in terms of coordinates $\bar{r}, \bar{\theta}$ defined by

$$\bar{r} = [r^2 - 2(ia/n)\cos\theta + (ia/n)^2]^{1/2} = (a/n)X, \tag{18}$$

$$\bar{r}\cos\bar{\theta} = r\cos\theta - (ia/n) = (a/n)(\hat{x}y - i). \tag{19}$$

Geometrically, $\bar{r}, \bar{\theta}$ are also spherical coordinates (see Fig. 3), but with their origin located at a point $z = ia/n$ on the symmetry axis, i.e., at $\hat{x}=i$, $y=1$. The multipole expansion of ξ about this point,

$$\xi_n = \sum C_{nl}\, \bar{r}^{-(l+1)} P_l(\cos\bar{\theta})$$

is quite simple, and Eqs. (14), (15), (17) show that it contains only terms up to $l=n$. Thus we have a model in which the source consists of *a finite number of point multipoles all placed at the same complex location* $z = ia/n$. It seems probable that the same type of model exists for higher values of n than $n=3$, but we have not explored this matter further.

At large radial distances one would like to have the linearized TS fields analyzed into their multipole moments. This is readily accomplished since Eqs. (14), (15), (17) can be expanded in spherical harmonics by means of a generating function. With the definition

$$\xi_n = \sum Q_{nl}\, r^{-(l+1)} P_l(\cos\theta)$$

we find that

$$Q_{1_l} = m(ia)^l,$$
$$Q_{2_l} = m(l+1)(ia/2)^l, \tag{20}$$
$$Q_{3_l} = \tfrac{1}{3}m(2l^2 + 4l + 3)(ia/3)^l.$$

These linearized moments are in agreement with the first few exact moments quoted in Eq. (12). One of us (W.K.) has also previously calculated[8] the exact multipole moments for the Kerr metric as far as $l=8$. This was done with the aid of SYMBAL, a formula-manipulating program available for the CDC6600 computer. To that point in the calculation, the amazing result was that *no term of order m^2 had appeared,* and the exact Janis moments were still reproducing exactly the linearized moments given in Eq. (20)! With the other TS metrics this is obviously not the case.

V. DISC MODELS

The attractive simplicity of the model discussed above is offset by its somewhat symbolic use of complex coordinates, and we therefore now consider an alternative Newtonian model with real mass and mass-current distributions. We have stated that the Ernst potential is singular at $\hat{x}=y=0$ which is a ring of radius a/n. Another singularity arises from the spheroidal coordinate system itself (see Fig. 2). The coordinate y is discontinuous across the entire disc $\hat{x}=0$, being positive on one face and negative on the other. This leads to a corresponding discontinuity in ξ, and a consequent necessity for sources everywhere on this surface.

On the disc $\hat{x}=0$, the radial coordinate is

$$\rho = (a/n)(1-y^2)^{1/2}, \tag{21}$$

the element of surface area is

$$|dA| = 2\pi\rho\, d\rho = 2\pi(a/n)^2 y\, dy, \tag{22}$$

and the normal derivative is

$$\frac{\partial}{\partial z} = \frac{n}{ay}\frac{\partial}{\partial x}. \tag{23}$$

On any surface $\hat{x} = \text{const} \neq 0$,

$$\xi_1 = iq(y + i\hat{x})^{-1},$$
$$\xi_2 = 2iq(y+i\hat{x})^{-1} + 2q\hat{x}(y+i\hat{x})^{-2} - 2iq(1+\hat{x})^2(y+i\hat{x})^{-3},$$
$$\xi_3 = 3iq(y+i\hat{x})^{-1} + 6q\hat{x}(y+i\hat{x})^{-2} - 4iq(2+3\hat{x}^2)(y+i\hat{x})^{-3} - 12q(\hat{x}+\hat{x}^3)(y+i\hat{x})^{-4} + 6iq(1+\hat{x}^2)^2(y+i\hat{x})^{-5}, \tag{24}$$

where $q = m/a$. As $\hat{x} \to 0$ we have

$$\xi_1 \to iqy_+^{-1},$$
$$\xi_2 \to iq(2y_+^{-1} - 2y_+^{-3}), \tag{25}$$
$$\xi_3 \to iq(3y_+^{-1} - 8y_+^{-3} + 6y_+^{-5}),$$

where

$$y_+^{-n} \equiv \lim_{\epsilon\to 0}(y + i\epsilon)^{-n}.$$

The functions y_+^{-n} must be understood as generalized functions,[9] and in that context they have nonvanishing imaginary parts. For example,

$$y_+^{-1} = y^{-1} - i\pi\delta(y).$$

All of the other functions may be obtained from this one by repeated differentiation.

In Newtonian gravity the mass density of a sheet is given by

$$\sigma = -(2\pi)^{-1}\mathbf{n}\cdot\nabla\Phi \tag{26}$$

and the current density is

$$\mathbf{j} = (2\pi)^{-1}\,\mathbf{n}\times\nabla\Omega, \tag{27}$$

where $\mathbf{n}$ is the unit normal. Using Eqs. (24), we find that

$$\begin{aligned}
\sigma_1 &= (m/2\pi a^2 y)(y^{-2}),\\
\sigma_2 &= (m/2\pi a^2 y)(2y^{-2} - 3y^{-4}),\\
\sigma_3 &= (m/2\pi a^2 y)(3y^{-2} - 12y^{-4} + 10y^{-6}).
\end{aligned} \tag{28}$$

The factor $1/y$ in all of these expressions has deliberately been isolated, for it must be eventually combined with the factor of y in the area element, Eq. (22). What remains in the parenthesis in σ_n is the generalized function. Note that σ_n is either positive definite or negative definite, depending on whether n is odd or even. Also note that as we approach the edge of the disc at $y=0$, σ_n diverges as

$$\sigma_n \sim [(a/n)^2 - \rho^2]^{-(2n+1)/2}. \tag{29}$$

This would appear to imply that the total mass,

$$M_n = \tfrac{1}{2}\int_{-1}^{1} 2\pi(a/n)\,\sigma_n\, y\, dy, \tag{30}$$

would have to be infinite. However, there is a further singularity in σ_n which is concentrated on the ring $y=0$ and which is due solely to its interpretation as a generalized function. The simple (but rigorous) rule for handling a divergent integral like M_n is that the expression is integrated and then evaluated at the end points $y=\pm 1$ *just as if no singularity at* $y=0$ *were present*. For all three cases we confirm in this manner that $M_n = m$. Roughly speaking, one may say that there is an infinite mass density residing on the ring, of such a sign and strength as to make the total mass of ring plus disc finite.

Now for all three values of n Eq. (25) shows that ξ_n is purely imaginary on the disc, and hence the surface is an equipotential. One might therefore wonder what relationship these solutions have to the familiar electrostatic problem of a charged conducting disc, in which the surface is also an equipotential. In that problem the solution is[10]

$$V = (2/\pi)\cot^{-1}\hat{x} = \frac{i}{\pi}\ln\left(\frac{\hat{x}-i}{\hat{x}+i}\right). \tag{31}$$

This function is singular at $\hat{x}=\pm i$, which is an entire line segment, $\rho=0$, $z=iay$, $-1\le y\le 1$. Moreover, σ, now a charge density, is once again divergent at the disc's edge,

$$\sigma = (a^2-\rho^2)^{-1/2}.$$

To examine the behavior one would generally expect to find there, let r, ϕ be a local set of cylindrical coordinates whose axis coincides with the edge. Laplace's equation in this neighborhood will have the solution

$$V \sim \sum r^m \cos m\phi.$$

Although m would normally be an integer, the presence of the disc forces the appropriate choice to be a half-(integer instead. The charged disc picks $m=\frac{1}{2}$, while for the family of TS solutions we have $m=-n+\frac{1}{2}$. (We might therefore hope that a TS solution will someday be discovered for $n=0$!) All of these solutions have period 4π in the angle ϕ, and it is therefore a quite natural thing to consider extending them to a twofold covering of Minkowski space using the ring as a branch line. This procedure is thus not a unique feature of the Kerr metric.

Returning to the mass-current densities, we find that

$$\begin{aligned}
j_1 &= -(m/2\pi a^2 y)(1-y^2)^{1/2}(y^{-2}),\\
j_2 &= -(4m/2\pi a^2 y)(1-y^2)^{1/2}(y^{-2} - 3y^{-4}),\\
j_3 &= -(9m/2\pi a^2 y)(1-y^2)^{1/2}(y^{-2} - 8y^{-4} + 10y^{-6}).
\end{aligned} \tag{32}$$

The total angular momentum is

$$J_n = \tfrac{1}{2}\int_{-1}^{1}\pi(a/n)^3 j_n\, y(1-y^2)^{1/2}\, dy, \tag{33}$$

which yields $J_n = ma$ for all three values of n. The velocity of rotation, even when special relativistic effects are included, is just

$$v_n = j_n/\sigma_n.$$

From Eqs. (28), (32) we see that the rotation is not rigid and that $v_n \to 1$ as the ring is approached.

VI. THE LIMIT $q=1$

To obtain the static (i.e., nonrotating) limit $a=0$ of the Kerr metric and the other TS metrics, we simply take the expression for the Ernst potential and set $p=1$, $q=0$. On the other hand, the so-called "extreme" Kerr limit $a=m$ *cannot* be obtained in so straightforward a manner merely by setting $p=0$, $q=1$. This is to say, the metric computed from $\xi^{-1}=-iy$ is *not* extreme Kerr. The reason that this limit needs special treatment may be seen in Eq. (8), where we observe that the transformation from (ρ,z) to (x,y) becomes singular as $p\to 0$. As a consequence there are various ways in which the limiting process might be performed, depending on whether ρ or x is required to remain finite. Possibly even some intermediate method might be attempted.

Consider, for example, the situation that arises for the Kerr metric itself. The relationship between Kerr coordinates R, Θ and the Weyl–Papapetrou coordinates is

$$\begin{aligned}
\rho &= (R^2 - 2mR + a^2)^{1/2}\sin\Theta\\
z &= (R-m)\cos\Theta
\end{aligned} \tag{34}$$

or alternatively

$$\begin{aligned}
mpx &= R-m\\
y &= \cos\Theta.
\end{aligned} \tag{35}$$

The Ernst potential is

$$\xi^{-1} = \frac{(R-m) - ia\cos\Theta}{m}. \tag{36}$$

Now, if the limit $p\to 0$ is taken holding either ρ or R finite, we obtain the usual extreme Kerr metric. On the other hand, if we allow our coordinates to be rescaled so that x remains finite and $\xi^{-1}=-iy$, then necessarily $\rho\to 0$ and $R\to m$. One might therefore presume that the metric we obtain in this manner would be the Kerr metric restricted to the null surface $R=m$, and hence a metric that lacks the full Lorentz signature. If ρ were strictly zero this would certainly be the case. In fact, the process only confines ρ to a neighborhood of the axis and we find that metric to be

$$\xi^{-1} = -iy = -i\cos\Theta,$$

$$ds^2 = \frac{\sin^2\Theta}{1+\cos^2\Theta}(dt - 2r\,d\varphi)^2 \tag{37}$$

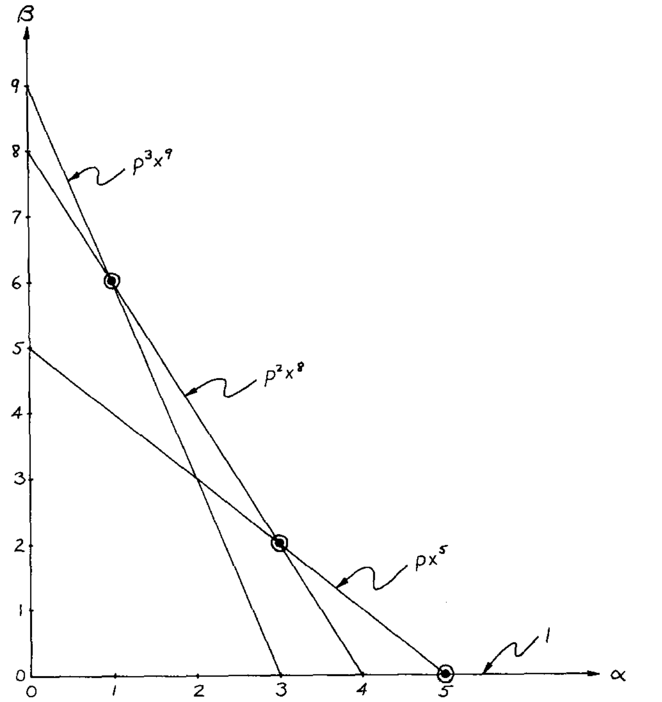

FIG. 4. Orders of magnitude of terms appearing in the Ernst potential for $n=3$. The vertical coordinate β is used to indicate that a term is $O(x^\beta)$. Bullseyes denote the distinguished limits at each odd value of α.

$$-(1+\cos^2\Theta)(r^{-2}dr^2+d\Theta^2+r^2d\varphi^2).$$

This is a type D metric and therefore a well-known one.[11,12] Its role as a limiting metric valid in the immediate vicinity of the extreme Kerr throat has been discussed previously by Bardeen.[13]

We next consider the situation that arises in the general TS metric. As Tomimatsu and Sato themselves have pointed out,[1] the limit may be taken holding the product px finite. If we then define R, Θ coordinates by

$$R-m=mpx/n, \quad \cos\Theta=y,$$

we will always obtain extreme Kerr in the limit, *regardless of which value of n we start with.*

Now we will show that there are other nontrivial ways of performing the limit that do lead to different solutions. Suppose the limit is taken in such a way that px^α remains finite, where α is an adjustable constant. The process is best illustrated using the case $n=3$. We write down from TS the Ernst potential, keeping only the leading terms in x:

$$\xi_3^{-1}=w/u,$$

$$w\approx p^3x^9-3ip^2x^8y-6px^5(1-y^4)+i(1-y^2)^3(y^3+3y), \tag{38}$$
$$u\approx 3p^2x^8-12ipx^5y(1-y^2)-(1-y^2)^3(3y^2+1).$$

The only powers of p and x that appear are p^3x^9, p^2x^8, px^5, and 1. In Fig. 4 we have plotted the order of magnitude of each term as a function of α. For almost all values of α, one term exceeds all the others in order of magnitude, and hence becomes the sole survivor as the limit is taken. For certain values, $\alpha=1, 3, 5$, the two largest terms happen to have the same order of magnitude, and we then obtain what is known as a "distinguished limit."

For $q=1$ limits which are not distinguished, the resulting Ernst potential must clearly be a function of y alone, and turns out to have the form

$$\xi^{-1}=-i\left(\frac{(1+y)^k-(1-y)^k}{(1+y)^k+(1-y)^k}\right), \tag{39}$$

where k is an integer, $k\leq n$. These solutions are not really new, since they can be quite easily obtained from the Voorhees metric[14] by making the replacement $x\longleftrightarrow y$. However, they may deserve more attention than has been previously been paid to them. The entire metric is

$$ds^2=f(dt-2kr\,d\varphi)^2$$
$$-f^{-1}[\sin^{2k^2}\Theta\, r^{-2k^2}(dr^2+r^2d\Theta^2)+r^2\sin^2\Theta d\varphi^2], \tag{40}$$

where f may be written as

$$f=2[\tan^{2k}(\theta/2)+\cot^{2k}(\theta/2)]^{-1}. \tag{41}$$

This solution is a type I generalization of Eq. (37) and describes a region of the TS metric near its ergosphere. Like Eq. (37), this metric is not asymptotically flat.

The distinguished limits lead to metrics which are apparently new ones. For example, for $\alpha=3$ we find

$$\xi^{-1}=-i\left(\frac{(1-y^4)+2ipx^3y}{2y(1-y^2)+2ipx^3}\right) \tag{42}$$

and this same solution is obtained for $\alpha=3$ from every TS metric regardless of which value of n we start with (provided only $n\geq 2$). The coordinates r, θ defined by

$$(r/m)^3=\tfrac{1}{2}px^3, \quad \cos\theta=y \tag{43}$$

will be spherical coordinates in the Weyl space, and ξ may be conveniently written in terms of them.

In general, for $\alpha=2k-1$ we define

$$\left(\frac{r}{m}\right)^{2k-1}=\frac{(k!)^2}{(2k)!}\,px^{2k-1}, \tag{44}$$
$$\cos\theta=y$$

and obtain the exact solution

$$\xi^{-1}=w/u,$$

$$w=(r/m)^{2k-1}[(1+y)^{k-1}-(1-y)^{k-1}]$$
$$-i(1-y^2)^{k-1}[(1+y)^k+(1-y)^k],$$

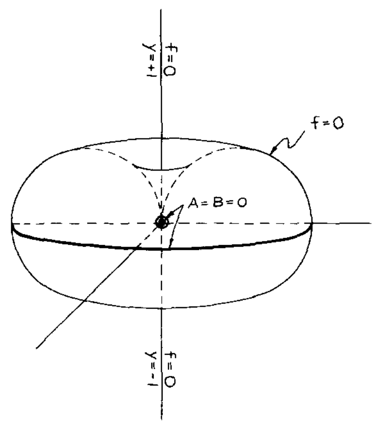

FIG. 5. Singularities of the function f in the new family of solutions, Eq. (46). Directional singularities occur at the origin and along the equator of the torus, as indicated.

$$u=(1-y^2)^{k-1}[(1+y)^k-(1-y)^k] + i(r/m)^{2k-1}[(1+y)^{k-1}+(1-y)^{k-1}]. \quad (45)$$

Furthermore, if Eq. (45) is now regarded simply as a solution in its own right (not derived from a TS solution), there is no reason to restrict k to be an integer, and we may allow it to take on any real value.

The metric has the form of Eq. (1) with

$$\begin{aligned} f &= 2(1-y^2)^{k-1}A/B, \\ \omega &= rC/mA, \\ \exp(2\gamma) &= A(1-y^2)^{(k-1)^2}(r/m)^{-2k^2} \end{aligned} \quad (46)$$

and

$$\begin{aligned} A &= (r/m)^{4k-2}-(1-y^2)^{2k-1}, \\ B &= (r/m)^{4k-2}[(1+y)^{2k-2}+(1-y)^{2k-2}] \\ &\quad -4(r/m)^{2k-1}(1-y^2)^{2k-2}+(1-y^2)^{2k-2}[(1+y)^{2k}+(1-y)^{2k}], \\ C &= 2(k-1)(r/m)^{4k-2}+(r/m)^{2k-1}[(1+y)^{2k-1} \\ &\quad +(1-y)^{2k-1}]-2k(1-y^2)^{2k-1}. \end{aligned} \quad (47)$$

These solutions have not yet been examined in any great detail, but we can make a few preliminary remarks about their properties. For either $r\to 0$ for $r\to\infty$ they asymptotically approach one or another of the undistinguished solutions of Eq. (42). Hence they are not asymptotically flat. The function f has zeroes on the symmetry axis $y=\pm 1$, and on a torus $(r/m)^2=1-y^2$ (see Fig. 5) and one would expect these to be surfaces of infinite redshift. However, the denominator B also vanishes at the origin $r=0$, $y=\pm 1$, and on the ring $r=m$, $y=0$. At these points f will possess an angular singularity similar to those which have already been discussed for the Weyl and TS metrics. For example, let

$$y=\epsilon, \quad r=m+\eta,$$

where ϵ, η are assumed small. Then we find, in a neighborhood of the ring,

$$f\approx\frac{1}{2k-1}\left(\frac{\epsilon^2+2\eta}{\epsilon^2+\eta^2}\right).$$

The limiting value of f will be infinite as long as we approach the ring along a straight path, $\epsilon/\eta=$ const, but if we approach it along a parabola $\epsilon^2/\eta=$ const, we can obtain a limit which is any finite value we please, including zero.

*Supported in part by NSF Grant GT-32157X.

[1]A. Tomimatsu and H. Sato, Prog. Theor. Phys. 50, 95 (1973).
[2]E. Glass, Phys. Rev. D 7, 3127 (1973).
[3]G. Gibbons and R. Russell-Clark, Phys. Rev. Lett. 30, 398 (1973).
[4]W. Israel, Phys. Rev. D 2, 641 (1970).
[5]F. Ernst, Phys. Rev. 168, 1415 (1968).
[6]Our Ernst potential is the reciprocal of the one introduced in Ref. 5. Equation (4) is invariant under $\xi\to\xi^{-1}$, and we have used this fact in order to have $\xi\to 0$ when $r\to\infty$.
[7]W. Kinnersley, J. Math. Phys. 14, 651 (1973).
[8]W. Kinnersley (unpublished).
[9]I.M. Gel'fand and G.E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I, p. 60. We have called y_+^{-n} what Gel'fand calls $(y+i0)^{-n}$.
[10]J. Jeans, *Mathematical Theory of Electricity and Magnetism* (Cambridge U.P., Cambridge, 1927), p. 249.
[11]B. Carter, Comm. Math. Phys. 10, 280 (1968).
[12]W. Kinnersley, J. Math. Phys. 10, 1195 (1969).
[13]J. Bardeen, unpublished communication to C. Misner.
[14]B. Voorhees, Phys. Rev. D 2, 2119 (1970).